Algebraic Geometry Lecture 36 – Arakelov Theory for Newbs

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1. Metrics on invertible sheaves

Let $\mathcal{L}$ be an invertible sheaf on a variety X over $\mathbb{C}$, i.e. a locally free sheaf of rank 1, so that for each open $U_i \subseteq \mathbb{C}$ we have $\mathcal{L}(U_i) \cong \mathcal{O}_X(U_i)$. Let

$$\varphi_i : \mathcal{L}(U_i) \longrightarrow \mathcal{O}_X(U_i)$$

be the isomorphisms and let the transition maps be

$$t_{ij} = \varphi_i \circ \varphi_j^{-1} : \mathcal{O}_X(U_i \cap U_j) \longrightarrow \mathcal{O}_X(U_i \cap U_j).$$

These are automorphisms and can be identified with multiplication by a nonzero element of $\mathcal{O}_X(U_i \cap U_j)$.

Let $s \in \mathcal{L}(U)$, then for a suitable choice of U_i, s corresponds via φ_i to an element $s_i \in \mathcal{O}_X(U \cap U_i)$. So

$$s_i = t_{ij} s_j$$

in $\mathcal{O}_X(U_i \cap U_j \cap U_k)$.

Dan showed us how to associate a sheaf $\mathcal{O}_X(D)$ to a Cartier divisor $D = (U_i, f_i)$ by letting $\mathcal{O}_X(D)(U_i)$ be generated by f_i^{-1} over $\mathcal{O}_X(U_i)$. Suppose we let $\mathcal{L} = \mathcal{O}_X(D)$ in the above, then we may choose φ_i to be multiplication by f_i.

A **metric** ρ on $\mathcal{L}$ is essentially a collection of norms, one on each fibre $\mathcal{L}_{(x)}$ for $x \in X$, and such that the norms vary as nicely as is expected of them. We define "varying nicely" using charts.

Suppose $\mathcal{L}$ is represented by $\{(U_i, t_{ij})\}$ and for each i we're given a function

$$\rho_i : U_i \longrightarrow \mathbb{R}_{>0}$$

such that on $U_i \cap U_j$ we have

$$\rho_i = |t_{ij}|^2 \rho_j,$$

then we say the family $\{(U_i, t_{ij}, \rho_i)\}$ **represents the metric**. The metric itself is either an equivalence class of such families or is the maximal family of compatible triples (in some ordering).

We've used the transition maps t_{ij} as part of the triple, but it's equally well defined if one uses the isomorphisms φ_i instead. An invertible sheaf $\mathcal{L}$ with a metric ρ will be denoted $\mathcal{L}_\rho$ or $(\mathcal{L}, \rho)$, and called a **metrized invertible sheaf**.

If $s \in \mathcal{L}(U_i)$ and $x \in U_i$ then we define

$$|s(x)|_\rho^2 = \frac{|s_i(x)|^2}{\rho_i(x)}.$$

Suppose also that $x \in U_j$, then

$$\frac{|s_i(x)|^2}{\rho_i(x)} = \frac{|\varphi_{ij} s_j(x)|^2}{|\varphi_{ij}|^2 \rho_j(x)} = \frac{|s_j(x)|^2}{\rho_j(x)},$$

so this is well defined.

For a given point x we may form the stalk $\mathcal{L}_x = \varinjlim_{U \ni x} \mathcal{L}(U)$, which is a one-dimensional free module over the local ring $\mathcal{O}_x$, and we let

$$\mathcal{L}_{(x)} = \mathcal{L}_x / \mathfrak{m}_x \mathcal{L}_x$$

(where $\mathfrak{m}_x$ is the maximal ideal – remember $\mathcal{L}(U_i) \cong \mathcal{O}_X(U_i)$) be the fibre at x. Then $\mathcal{L}_{(x)}$ is a one-dimensional vector space over $\mathbb{C}$. A metric ρ on $\mathcal{L}$ induces a norm on $\mathcal{L}_{(x)}$, for if $s \in \mathfrak{m}_x \mathcal{L}_x$, so $s(x) = 0$, then $|s_i(x)| = 0$. So for $s \in \mathcal{L}_x$, $|s(x)|_\rho$ only depends on $s \pmod{\mathfrak{m}_x \mathcal{L}_x}$, thus giving a norm on the fibre.

2. Example: the standard metric

Let $X = \mathbb{P}^n_{\mathbb{C}}$ with variables $x_0, \ldots, x_n$. Let $U_i = \{x_i \neq 0\}$ and let

$$z_0^{(i)} = \frac{x_0}{x_i}, \quad z_1^{(i)} = \frac{x_1}{x_i}, \quad \ldots \quad , \quad z_n^{(i)} = \frac{x_n}{x_i}$$

be coordinates on U_i along with the constant 1. Define functions $\rho_i : U_i \to \mathbb{R}_{>0}$ by

$$\rho_i(z) = \sum_{j=0}^{n} |z_j^{(i)}|^2 = \sum_{j=0}^{n} |x_j/x_i|^2.$$

On $U_i \cap U_j$ we have

$$\rho_i = |x_j/x_i|^2 \rho_j,$$

so the family

$$\{(U_i, x_j/x_i, \rho_i)\} = \{(\{x_i \neq 0\}, z_j^{(i)}, \sum_{j=0}^{n} |z_j^{(i)}|^2)\}$$

is a metric on the invertible sheaf represented by $\{(U_i, x_j/x_i)\}$, which as Dan showed us last week corresponds to the line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$.

3. Weil functions

Let $D = (U, f)$ be a Cartier divisor on X. A **Weil function** λ_D associated with D is a function

$$\lambda_D : X \setminus \operatorname{supp}(D) \longrightarrow \mathbb{R}$$

($\operatorname{supp}(D) = \{x \in X \ : \ f(x) = 0, \text{ or } 1/f(x) = 0\}$), such that there is a continuous function $\alpha : U \to \mathbb{R}$ with

$$\lambda_D(x) = \log |f(x)| + \alpha(x), \qquad x \in U \setminus \operatorname{supp}(D).$$

D is called the divisor of λ_D.

Example. Let $X = \mathbb{P}^1_{\mathbb{C}}$ and $D = (\infty)$. Let z be our complex variable, then

$$\lambda(z) = \log \max\{1, |x|\}$$

is a Weil function associated with D.

If $\mathcal{L}_\rho$ is a metrized invertible sheaf and s is a rational section with divisor D, then we define the Weil function associated with the metric and s by

$$\lambda_{s,\rho}(x) = -\log |s(x)|_\rho, \qquad x \notin \operatorname{supp}(D).$$

In fact we can sometimes go the other way too. Specifically we have the following.

Proposition. *There is a bijection between Weil functions whose divisor is D and metrics on the invertible sheaf $\mathcal{O}_X(D)$.*